# Determinants

Last Time: Computational Introduction to Determinants.

↳ Cofactor Expansion Formula
(AKA Laplace Expansion Formula)

↳ Many Examples...

↳ Determinants of Elementary Matrices. *

Recall: $\det(P_{i,j}) = -1 \qquad (i \neq j)$

$\det(M_i(k)) = k$

$\det(A_{i,j}(k)) = 1$

Def$^n$: The $n \times n$ determinant function is the function $\det : \mathcal{M}_{n\times n} \to \mathbb{R}$ satisfying these conditions:

1. $\det(\ell_1, \ell_2, \cdots, k\ell_i + \ell_j, \cdots, \ell_n) = \det(\ell_1, \ell_2, \cdots, \ell_n)$.
2. $\det(\ell_1, \ell_2, \cdots, \ell_{i-1}, \ell_j, \ell_{i+1}, \cdots, \ell_{j-1}, \ell_i, \ell_{j+1}, \cdots, \ell_n) = -\det(\ell_1, \ell_2, \cdots, \ell_n)$.
3. $\det(\ell_1, \ell_2, \cdots, k\ell_i, \cdots, \ell_n) = k\det(\ell_1, \cdots, \ell_n)$
4. $\det(I_n) = 1$.

**NB:** The above properties are indeed satisfied by the Cofactor Expansion formula... (they're a lot nasty to prove..)

**Point:** determinants are computable using row operations :)

**Ex:** Compute $\det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 3 & 0 & 1 & -5 \\ 1 & 2 & 3 & 5 \\ 5 & 10 & 15 & 20 \end{bmatrix}$.

**Sol:**

$$\det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 3 & 0 & 1 & -5 \\ 1 & 2 & 3 & 5 \\ 5 & 10 & 15 & 20 \end{bmatrix} \quad \star$$

$$= \underline{5}\det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 3 & 0 & 1 & -5 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

subtracting multiples of $\ell_1$ from $\ell_2, \ell_3, \ell_4$

$$= 5\det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 0 & 4 & -14 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

swap

$$= 5\cdot(-1)\det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & -14 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

$$= -5\det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & -14 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

row-echelon form, "eliminate upwards"

$$= -5\det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

apply scaling rule multiple times..

$$= -5(2)(4)(-1)\det(I_4) = -5\cdot 2\cdot 4\cdot -1\cdot 1 = \underline{40} \quad \boxtimes$$

**Exercise:** Compute $\det(M)$ above via cofactor expansion...

Ex: Compute $\det\begin{bmatrix}-1 & 1 & 5\\ 5 & 8 & 6\\ 3 & 9 & -7\end{bmatrix}$.

Sol: $\det\begin{bmatrix}-1 & 1 & 5\\ 5 & 8 & 6\\ 3 & 9 & -7\end{bmatrix} = \det\begin{bmatrix}-1 & 1 & 5\\ 0 & 13 & 31\\ 0 & 12 & 8\end{bmatrix}$

$$= 4\det\begin{bmatrix}-1 & 1 & 5\\ 0 & 13 & 31\\ 0 & 3 & 2\end{bmatrix}$$

$$= 4\cdot\frac{1}{3}\det\begin{bmatrix}-1 & 1 & 5\\ 0 & \boxed{39} & 93\\ 0 & 3 & 2\end{bmatrix}$$

Echelon form.

$$= \frac{4}{3}\det\begin{bmatrix}-1 & 1 & 5\\ 0 & \boxed{0} & 67\\ 0 & 3 & 2\end{bmatrix} = -\frac{4}{3}\det\begin{bmatrix}-1 & 1 & 5\\ 0 & 3 & 2\\ 0 & 0 & 67\end{bmatrix}$$

$$= -\frac{4}{3}\begin{bmatrix}-1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 67\end{bmatrix} = -\frac{4}{3}(-1)(3)(67)\det\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= -\frac{4}{3}(-1)(3)(67)\cdot 1 = 268 \checkmark \qquad \square$$

Sol 2 (Via Cofactor Expansion):

$$\det\begin{array}{c}\begin{array}{ccc}+ & - & +\end{array}\\ \begin{bmatrix}-1 & 1 & 5\\ 5 & 8 & 6\\ 3 & 9 & -7\end{bmatrix}\end{array} = -\det\begin{bmatrix}8 & 6\\ 9 & -7\end{bmatrix} - \det\begin{bmatrix}5 & 6\\ 3 & -7\end{bmatrix} + 5\det\begin{bmatrix}5 & 8\\ 3 & 9\end{bmatrix}$$

$$= -(8\cdot -7 - 6\cdot 9) - (5\cdot -7 - 6\cdot 3) + 5(5\cdot 9 - 3\cdot 8)$$

$$= -(-56 - 54) - (-35 - 18) + 5(45 - 24)$$

$$= 110 + 53 + 5(21) = 163 + 105 = 268 \qquad \square$$

**Prop:** The cofactor Expansion Formula and the properties of det given at the beginning of the lecture determine the same quantity for every $n\times n$ matrix. In particular, the determinant function is given by either.

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**Ex:** Compute $\det\begin{bmatrix}1&2&1\\-1&1&3\\0&3&4\end{bmatrix}$

**Sol:**

$$\det\begin{bmatrix}1&2&1\\-1&1&3\\0&3&4\end{bmatrix}$$

$$=\det\begin{bmatrix}1&2&1\\0&3&4\\0&3&4\end{bmatrix}$$

$$=\det\begin{bmatrix}1&2&1\\0&3&4\\0&0&0\end{bmatrix}=0$$

$\boxed{/\!/}$

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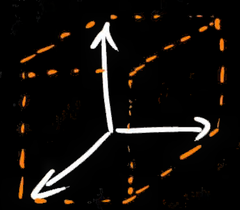
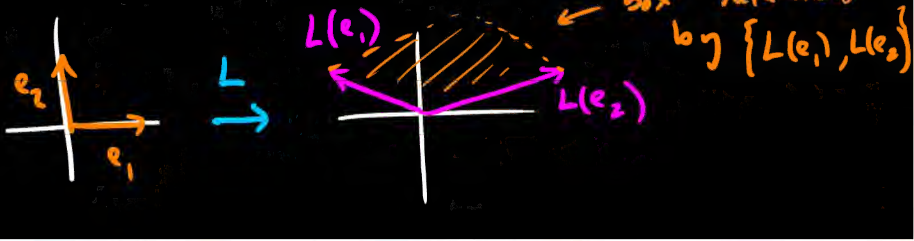
**Prop:** Let $L:\mathbb{R}^n\to\mathbb{R}^n$ be a linear transformation. Let $[L]$ be the matrix of $L$ with respect to the standard basis on $\mathbb{R}^n$ (i.e. $[L]=[L(e_1)\,|\,L(e_2)\,|\cdots|\,L(e_n)]$).

$\mathcal{E}_n$



The determinant $\det[L]$ is the "signed volume" of the "box" determined by $\{L(e_1),L(e_2),\dots,L(e_n)\}$.

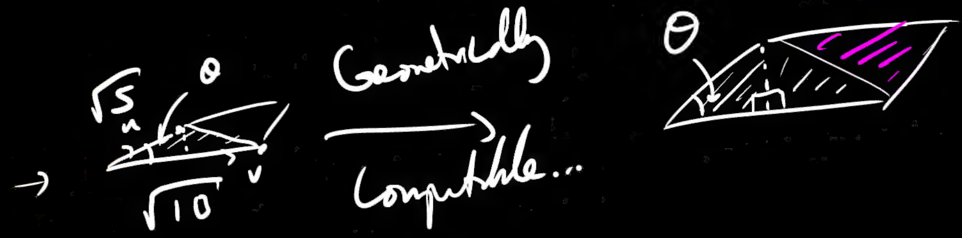
Picture in $\mathbb{R}^2$:



NB: Proof omitted for time, see Hefferon... $\boxed{/\!/}$

Ex: Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ have matrix $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$.

"Box" = "parallelopiped".


$$\det \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = 2 - 3 = -1 \Leftarrow \therefore \text{Area} = |-1| = 1. \quad \boxed{/\!/}$$

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Cor: The determinant is multiplicative. I.E. For $A, B \in M_{n \times n}$ we have $\det(AB) = \det(A)\det(B)$.

Pf: $A$ and $B$ determine two linear transformations $\mathbb{R}^n \to \mathbb{R}^n$. The product is the matrix of their composition. Then $\det(AB) \overset{*}{=}$ volume of the parallelopiped determined by $AB(E_n) = A(BE_n)$. So we see

$$\det(AB) \overset{*}{=} \det(A) \cdot \text{volume}(\text{parallelopiped given by } BE_n)$$

$$\overset{*}{=} \det(A)\det(B). \quad \boxed{/\!/}$$

Proposition

NB: This isn't particularly surprising... The definition of the determinant given today encodes the conditions

$$\det(\text{Product of elem mats}) = \text{prod}(\text{dets of the elem mats})$$

Cor: Suppose $A$ is invertible. Then $\det(A^{-1}) = \frac{1}{\det(A)}$.

pf: If $A$ is invertible, then $I_n = A^{-1}A$,

so $1 = \det(I_n) = \det(A^{-1}A) = \det(A^{-1}) \cdot \det(A)$.

Hence dividing both sides by $\det(A)$ yields result. $\boxtimes$

Exercise: Check for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ directly ...

Cor: Let $A$ be an $n \times n$ matrix. Then $\det(A) \neq 0$ if and only if $A$ is invertible.

pf: If $A$ is invertible, $\det(A^{-1}) \cdot \det(A) \neq 0$, so $\det(A) \neq 0$.

If $\det(A) \neq 0$, then $L_A : \mathbb{R}^n \to \mathbb{R}^n$ determined by $A$ takes the parallelopiped of $\mathcal{E}_n$ to a parallelopiped of nonzero volume. Moreover, if $L_A(x) = 0$ for $x \neq 0$, then extending $\{x\}$ to a basis of $\mathbb{R}^n$ would yield a parallelopiped which maps under $L_A$ to a zero-volume parallelopiped, hence contradicting the theorem. $\boxtimes$